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ASYMPTOTIC GREEDINESS OF THE HAAR SYSTEM IN THE SPACES $L_p[0,1]$, $1<p<\infty$

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Abstract. Our aim in this paper is to attain a sharp asymptotic estimate for the greedy constant $C_g[\mathcal{H}^{(p)}, L_p]$ of the (normalized) Haar system $\mathcal{H}^{(p)}$ in $L_p[0,1]$ for $1<p<\infty$. We will show that the superdemocracy constant of $\mathcal{H}^{(p)}$ in $L_p[0,1]$ grows as $p^*=\max\{p, p/(p-1)\}$ as p^* goes to ∞. Thus, since the unconditionality constant of $\mathcal{H}^{(p)}$ in $L_p[0,1]$ is p^*-1, the well-known general estimates for the greedy constant of a greedy basis obtained from the intrinsic features of greediness (namely, democracy and unconditionality) yield that $p^* \lesssim C_g[\mathcal{H}^{(p)}, L_p] \lesssim (p^*)^2$. Going further, we develop techniques that allow us to close the gap between those two bounds, establishing that, in fact, $C_g[\mathcal{H}^{(p)}, L_p] \approx p^*$. Our work answers a question that was raised by T. Hytonen [9].

1. Introduction

A fundamental and total biorthogonal system for an infinite-dimensional separable Banach space $(\mathbb{X}, \|\cdot\|)$ over the field $\mathbb{F}$ of real or complex scalars, is a family $(\mathbf{x}_j, \mathbf{x}_j^*)_{j\in J}$ in $\mathbb{X}\times\mathbb{X}^*$ verifying

(i) $\mathbb{X} = \overline{\operatorname{span}\{\mathbf{x}_j : j\in J\}}$,
(ii) $\mathbb{X}^* = \overline{\operatorname{span}\{\mathbf{x}_j^* : j\in J\}}^{w^*}$, and
(iii) $\mathbf{x}_j^*(\mathbf{x}_k)=1$ if $j=k$ and $\mathbf{x}_j^*(\mathbf{x}_k)=0$ otherwise.

The family $\mathcal{B}=(\mathbf{x}_j)_{j\in J}$ is called a *(Markushevich) basis* and the unequivocally determined collection of bounded linear functionals $\mathcal{B}^* = (\mathbf{x}_j^*)_{j\in J}$ is said to be the family of *coordinate functionals* (or *dual basis*) of $\mathcal{B}$. If the biorthogonal system verifies the condition

(iv) $\sup_{j\in J}\|\mathbf{x}_j\|\|\mathbf{x}_j^*\|<\infty$

we say that $\mathcal{B}$ is *M-bounded.* Finally, if we have

(v) $0<\inf_{j\in J}\|\mathbf{x}_j\| \le \sup_{j\in J}\|\mathbf{x}_j\|<\infty$

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we say that $\mathcal{B}$ is *semi-normalized* (*normalized* if $\|\mathbf{x}_j\| = 1$ for all $j \in J$). Note that a basis $\mathcal{B}$ verifies simultaneously (iv) and (v) if and only if

$$\sup_{j\in J} \max\{\|\mathbf{x}_j\|, \|\mathbf{x}_j^*\|\} < \infty.$$

Suppose $\mathcal{B} = (\mathbf{x}_j)_{j\in J}$ is a semi-normalized M-bounded basis in a Banach space $\mathbb{X}$ with coordinate functionals $(\mathbf{x}_j^*)_{j\in J}$. Then each $f \in \mathbb{X}$ is uniquely determined by its *coefficient family* $(\mathbf{x}_j^*(f))_{j\in J}$, which belongs to $c_0(J)$. Thus, we can consider its non-increasing rearrangement, which we denote by $(a_m^*[\mathcal{B}, \mathbb{X}](f))_{m=1}^\infty$ (or, simply $(a_m^*(f))_{m=1}^\infty$ if the basis and the space are clear from context).

For each $f \in \mathbb{X}$ there is a one-to-one map $\rho\colon \mathbb{N} \to J$ such that

$$|\mathbf{x}_{\rho(m)}^*(f)| = a_m^*(f), \quad m \in \mathbb{N}. \tag{1.1}$$

If the family $(\mathbf{x}_j^*(f))_{j\in J}$ contains several terms with the same absolute value then the map ρ for f is not uniquely determined. In order to get uniqueness we arrange the countable set J by means of a bijection $\sigma\colon J \to \mathbb{N}$ and impose the additional condition

$$\sigma(\rho(m)) \le \sigma(\rho(n)) \text{ whenever } |\mathbf{x}_{\rho(m)}^*(f)| = |\mathbf{x}_{\rho(n)}^*(f)| \text{ and } m \le n. \tag{1.2}$$

If f is supported on an infinite set there is a unique one-to-one map $\rho\colon \mathbb{N} \to J$ with $\rho(\mathbb{N}) = \operatorname{supp}(f)$ that verifies (1.1) and (1.2). In the case when f is finitely supported, there is a unique bijection $\rho\colon \mathbb{N} \to J$ that verifies (1.1) and (1.2). In any case, we will refer to such a map ρ as the *greedy ordering* for f. For each $m \in \mathbb{N}$, the *mth-greedy approximation* to f is the partial sum

$$\mathcal{G}_m[\mathcal{B}, \mathbb{X}](f) := \mathcal{G}_m(f) = \sum_{n=1}^m \mathbf{x}_{\rho(n)}^*(f)\, \mathbf{x}_{\rho(n)},$$

where ρ is the greedy ordering for f. The sequence $(\mathcal{G}_m(f))_{m=1}^\infty$ is called the *greedy algorithm* for f with respect to the basis $\mathcal{B}$.

To quantify the efficiency of the greedy algorithm, Temlyakov [12] introduced the sequence $(\mathbf{L}_m)_{m=1}^\infty$ of *Lebesgue greedy constants*. For each $m \in \mathbb{N}$, $\mathbf{L}_m[\mathcal{B}, \mathbb{X}] := \mathbf{L}_m$ is the smallest constant C such that the Lebesgue-type inequality

$$\|f - \mathcal{G}_m(f)\| \le C \left\| f - \sum_{j\in A} a_j\, \mathbf{x}_j \right\|, \tag{1.3}$$

holds for all $f \in \mathbb{X}$, all subsets A of $\mathbb{N}$ with $|A| = m$, and all $a_j \in \mathbb{F}$. Konyagin and Temlyakov [10] then defined a basis $\mathcal{B}$ to be *greedy* if (1.3) holds with a constant $1 \le C < \infty$ independent of m. The smallest admissible constant C will be denoted by $C_g[\mathcal{B}, \mathbb{X}] = C_g$, and will be referred to as the *greedy constant* of the basis. In other words, a basis $\mathcal{B}$ is greedy if and only if $\sup_m \mathbf{L}_m = C_g < \infty$, that is, the greedy algorithm provides, up to a multiplicative constant, the best m-term approximation to any vector in the space.

Once we know that a certain basis is greedy, a natural problem in approximation theory is to compute, or at least estimate, its greedy constant. Also of interest is to determine for what values of C a basis is C-greedy under a suitable renorming of the space.

In this paper we focus on the Haar system $\mathcal{H}^{(p)} = (h_I^{(p)})_{I\in\mathcal{D}_0}$ in the spaces $L_p[0,1]$ (L_p for short from now on). Here, $\mathcal{D}_0$ denotes the set $\mathcal{D}\cup\{0\}$, where $\mathcal{D}$ is the collection of all dyadic intervals contained in $[0,1)$, $h_0^{(p)}$ is the constant function 1 on $[0,1)$, and $h_I^{(p)}$ stands for the L_p-normalized Haar function supported on I, i.e., if $I=[a,b)$ then

$$h_I^{(p)}(t) = \begin{cases} -2^{-(b-a)/p} & \text{if } a \le t < (a+b)/2, \\ 2^{-(b-a)/p} & \text{if } (a+b)/2 \le t < b. \end{cases}$$

Thus, $\mathcal{H}^{(p)}$ is a normalized M-bounded basis for L_p when $1\le p<\infty$, whereas $\mathcal{H}^{(\infty)}$ is a normalized M-bounded basis for the subspace $\mathbb{D}$ that it generates in L_∞. In either case, the family of coordinate functionals of $\mathcal{H}^{(p)}$ is $\mathcal{H}^{(p')}$ (with the canonical isometric identification of functions in $L_{p'}$ with functionals in $(L_p)^*$, where $p'=p/(p-1)$). In fact, when arranged in the natural way, $\mathcal{H}^{(p)}$ is a Schauder basis (see [2, Proposition 6.1.3]).

Temlyakov [11] showed that $\mathcal{H}^{(p)}$ is a greedy basis in L_p for $1<p<\infty$. Later on, Dilworth et al. [7] proved that for every $C>1$ there is a renorming of L_p with respect to which $\mathcal{H}^{(p)}$ is C-greedy. Whether or not the isometric constant $C=1$ can be achieved up to renorming (see [3]) remains unknown as of today.

Note that, since the Haar system is not an unconditional basis in L_1 (see [2, Proposition 6.3.1]), neither $\mathcal{H}^{(1)}$ is a greedy basis for L_1 nor $\mathcal{H}^{(\infty)}$ is a greedy basis for $\mathbb{D}$. Consequently we have

$$\lim_{p\to 1^+} C_g[\mathcal{H}^{(p)}, L_p] = \infty = \lim_{p\to\infty} C_g[\mathcal{H}^{(p)}, L_p].$$

The Haar system $\mathcal{H}^{(2)}$ is an orthonormal basis for L_2, which easily yields $C_g[\mathcal{H}^{(2)}, L_2]=1$. However, for $p\neq 2$ it seems hopeless to attempt to compute the exact value of $C_g[\mathcal{H}^{(p)}, L_p]$. It is therefore natural to address the problem of obtaining asymptotic estimates for $C_g[\mathcal{H}^{(p)}, L_p]$ as p tends to 1 or to ∞.

A standard approach to estimate the greedy constant of a greedy basis is to make use of its intrinsic properties instead of relying on the mere definition. The first movers in this direction were Konyagin and Temlyakov [10], who proved that a basis is greedy if and only if it is *unconditional* and *democratic*. To set the notation, we recall that a basis $\mathcal{B} = (\mathbf{x}_j)_{j\in J}$ for a Banach space $\mathbb{X}$ is said to be unconditional if the series expansion $\sum_{j\in J} \mathbf{x}_j^*(f)\, \mathbf{x}_j$ converges unconditionally to f for every $f \in \mathbb{X}$. Unconditional bases are characterized as those bases verifying the uniform bound

$$K_{su}[\mathcal{B}, \mathbb{X}] := \sup_{\substack{A\subset J \\ |A|<\infty}} \|S_A\| = \sup_{\substack{A\subset J \\ |A|<\infty}} \|\mathrm{Id}_{\mathbb{X}} - S_A\| < \infty, \tag{1.4}$$

or, equivalently,

$$K_u[\mathcal{B}, \mathbb{X}] := \sup\{\|M_\varepsilon\| \colon \varepsilon = (\varepsilon_j)_{j\in J},\, |\varepsilon_j| = 1\} < \infty, \tag{1.5}$$

where $S_A = S_A[\mathcal{B}, \mathbb{X}]$ is the coordinate projection on a finite set $A \subseteq J$, i.e.,

$$S_A \colon \mathbb{X} \to \mathbb{X}, \quad f \mapsto \textstyle\sum_{j\in A} \mathbf{x}_j^*(f)\, \mathbf{x}_j,$$

and $M_\varepsilon = M_\varepsilon[\mathcal{B}, \mathbb{X}]$ is the linear operator from $\mathbb{X}$ into $\mathbb{X}$ given by $\mathbf{x}_j \mapsto \varepsilon_j\, \mathbf{x}_j$. The *suppression unconditional constant* K_{su} and the *lattice unconditional constant* K_u of a basis are related by the inequalities

$$K_{su}[\mathcal{B}, \mathbb{X}] \le K_u[\mathcal{B}, \mathbb{X}] \le \kappa\, K_{su}[\mathcal{B}, \mathbb{X}], \tag{1.6}$$

where $\kappa = 2$ if $\mathbb{F} = \mathbb{R}$ and $\kappa = 4$ if $\mathbb{F} = \mathbb{C}$. In turn, a basis $\mathcal{B}$ is said to be *democratic* if there is $1 \le C < \infty$ such that

$$\left\| \sum_{j\in A} \mathbf{x}_j \right\| \le C \left\| \sum_{j\in B} \mathbf{x}_j \right\|, \quad |A| = |B| < \infty. \tag{1.7}$$

We will denote by $\Delta[\mathcal{B}, \mathbb{X}]$ the optimal constant C in (1.7). By imposing the extra assumption $A \cap B = \emptyset$ in (1.7) we obtain an equivalent definition of democracy, and $\Delta_d[\mathcal{B}, \mathbb{X}]$ will denote the optimal constant in (1.7) under the extra assumption on disjointness of sets.

Amalgamating some steps in Konyagin–Temlyakov's proof (see [7, Equation 1]) we get the estimate

$$\max\{K_{su}[\mathcal{B}, \mathbb{X}], \Delta[\mathcal{B}, \mathbb{X}]\} \le C_g[\mathcal{B}, \mathbb{X}] \le K_{su}[\mathcal{B}, \mathbb{X}] + \Delta_d[\mathcal{B}, \mathbb{X}] K_u^2[\mathcal{B}, \mathbb{X}]. \tag{1.8}$$

Note that

$$\Delta[\mathcal{B}, \mathbb{X}] \le K_{su}[\mathcal{B}, \mathbb{X}](1 + \Delta_d[\mathcal{B}, \mathbb{X}]).$$

Hence, when $\mathcal{B}$ runs over a certain family of bases, the left-hand side and the right-hand side terms in inequality (1.8) are of the same order if and only if the constants $K_u[\mathcal{B}, \mathbb{X}]$ are uniformly bounded. This is not the case for $(\mathcal{H}^{(p)})_{p>1}$ as the classical theorem of Burkholder shows.

Theorem 1.1 ([5])**.** *If $1 < p < \infty$ then $K_u[\mathcal{H}^{(p)}, L_p] = p^* - 1$, where $p^* = \max\{p, p'\}$.*

Now, we may try to obtain estimates that bring us closer to our goal by using super-democracy instead of democracy. A basis is *super-democratic* if there is a constant $1 \le C < \infty$ such that

$$\left\| \sum_{j\in A} \varepsilon_j \, \mathbf{x}_j \right\| \le C \left\| \sum_{j\in B} \delta_j \, \mathbf{x}_j \right\|, \quad |A| = |B| < \infty, \, |\varepsilon_j| = |\delta_j| = 1. \tag{1.9}$$

The smallest admissible constant in (1.9) will be denoted by $\Delta_s[\mathcal{B}, \mathbb{X}]$, and the smallest constant in (1.9) with the extra assumption $A \cap B = \emptyset$ will be denoted by $\Delta_{sd}[\mathcal{B}, \mathbb{X}]$.

Bases that are unconditional and democratic are super-democratic. Hence, greedy bases are characterized as those that are simultaneously unconditional and super-democratic. Quantitatively, a slight improvement of the argument used in the proof of [4, Theorem 1.3] gives

$$\max\{K_{su}[\mathcal{B}, \mathbb{X}], \Delta_{sd}[\mathcal{B}, \mathbb{X}]\} \le C_g[\mathcal{B}, \mathbb{X}] \le K_{su}[\mathcal{B}, \mathbb{X}](1 + \Delta_{sd}[\mathcal{B}, \mathbb{X}]).$$

These inequalities allow us to determine the rate of growth of the constants $C_g[\mathcal{B}, \mathbb{X}]$ when $\mathcal{B}$ runs over a family of bases only when $\min\{[K_{su}[\mathcal{B}, \mathbb{X}], \Delta_{sd}[\mathcal{B}, \mathbb{X}]\}$ is uniformly bounded. But, again, this is not the case for the L_p-normalized Haar system for $1 < p < \infty$. Indeed, on the one hand we have $\Delta_{sd}[\mathcal{B}, \mathbb{X}] \ge \Delta_d[\mathcal{B}, \mathbb{X}]$ for any $\mathcal{B}$ and any $\mathbb{X}$. On the other hand, the following result (which we shall prove below) yields $\sup_{p>1} \Delta_d[\mathcal{H}^{(p)}, L_p] = \infty$.

Proposition 1.2. *If $1 < p < \infty$ then*

$$\Delta_d[\mathcal{H}^{(p)}, L_p] \ge d_p := \frac{2^{1/p^\#} - 1}{2^{1/p^*} - 1},$$

where $p^\# = \min\{p, p'\}$.

Another important property of bases that comes into play in this scenario is the *symmetry for largest coefficients* (a.k.a. *Property A*). A basis $\mathcal{B}$ is said to be symmetric for largest coefficients is there is a constant $1 \le C < \infty$ such that

$$\left\| \sum_{j\in A} \varepsilon_j \, \mathbf{x}_j + f \right\| \le C \left\| \sum_{j\in B} \delta_j \, \mathbf{x}_j + f \right\| \tag{1.10}$$

whenever $|A| = |B| < \infty$, $A \cap B = (A \cup B) \cap \operatorname{supp}(f) = \emptyset$, and $|\mathbf{x}_i^*(f)| \le |\varepsilon_j| = |\delta_k| = 1$ for all $i \in J$, $j \in A$, $k \in B$. We denote by $C_a[\mathcal{B}, \mathbb{X}]$ the optimal constant C in (1.10). A basis is greedy if and only if it is unconditional and symmetric for largest coefficients. Moreover (see [1, Remark 3.8]),

$$\max\{K_{su}[\mathcal{B},\mathbb{X}], C_a[\mathcal{B},\mathbb{X}]\} \le C_g[\mathcal{B},\mathbb{X}] \le C_a[\mathcal{B},\mathbb{X}] K_{su}[\mathcal{B},\mathbb{X}]. \tag{1.11}$$

These estimates are useful when one wants to show that the greedy constant of a certain basis is close to 1. However, since

$$C_a[\mathcal{B},\mathbb{X}] \ge \Delta_{sd}[\mathcal{B},\mathbb{X}], \tag{1.12}$$

and the side terms of (1.11) are of different order, they do not provide a tight information about the asymptotic growth of the greedy constants of a family of bases.

Despite the fact that the methods described above are not strong enough to be applied to our problem, in this note we shall reach our goal and prove the following theorem.

Theorem 1.3. *$C_g[\mathcal{H}^{(p)}, L_p] \approx p^*$ for $1 < p < \infty$.*

The key idea in the proof of Theorem 1.3 will consist of taking advantage of the fact that the Haar system in L_p belongs to a more demanding class of bases than that of greedy bases, namely, the class of *bi-greedy* bases. A basis is said to be bi-greedy if both the basis itself and its dual basis are greedy. Bi-greedy bases were characterized in [6] as those bases that are unconditional and *bi-democratic*. Recall that a basis $\mathcal{B} = (\mathbf{x}_j)_{j=1}^\infty$ is said to be bi-democratic if there is a constant $1 \le C < \infty$ such that

$$\left\| \sum_{j\in A} \mathbf{x}_j \right\| \left\| \sum_{k\in B} \mathbf{x}_k^* \right\| \le Cm, \quad |A| = |B| = m. \tag{1.13}$$

We will denote by $\Delta_b[\mathcal{B},\mathbb{X}]$ the smallest constant C such that (1.13) holds, and we will refer to it as the bi-democratic constant of the basis.

The following new estimate for the greedy constant will also be crucial in the proof of Theorem 1.3.

Theorem 1.4. *Let $\mathcal{B}$ be a bi-democratic and unconditional basis for a Banach space $\mathbb{X}$. Then*

$$C_g[\mathcal{B},\mathbb{X}] \le K_{su}[\mathcal{B},\mathbb{X}] + \kappa^2 \, \Delta_b[\mathcal{B},\mathbb{X}].$$

Section 2 is devoted to proving Theorem 1.4, while in Section 3 we obtain the remaining estimates that in combination with Theorem 1.4 will yield Theorem 1.3. For the convenience of the reader, we have included as an appendix in Section 4 a summary of the constants related to greedy-like properties that are most commonly employed.

Throughout this article we use standard facts and notation from Banach spaces and approximation theory, as can be found, e.g., in [2].

Here, and throughout this paper, the symbol $\alpha_i \lesssim \beta_i$ for $i \in I$ means that the families of positive real numbers $(\alpha_i)_{i\in I}$ and $(\beta_i)_{i\in I}$ verify $\sup_{i\in I} \alpha_i/\beta_i < \infty$. If $\alpha_i \lesssim \beta_i$ and $\beta_i \lesssim \alpha_i$ for $i \in I$ we say $(\alpha_i)_{i\in I}$ are $(\beta_i)_{i\in I}$ are equivalent, and we write $\alpha_i \approx \beta_i$ for $i \in I$.

2. A new estimate for the Lebesgue type constants for the greedy algorithm using bi-democracy

The *fundamental function* of a basis $\mathcal{B} = (\mathbf{x}_j)_{j=1}^\infty$ in a Banach space $\mathbb{X}$ is the sequence given by

$$\varphi_m[\mathcal{B}, \mathbb{X}] = \sup_{|A|\le m} \left\| \sum_{j\in A} \mathbf{x}_j \right\|, \quad m \in \mathbb{N}.$$

We shall also consider the *super-fundamental function* of the basis, given by

$$\varphi_m^\varepsilon[\mathcal{B}, \mathbb{X}] = \varphi_m^\varepsilon := \sup_{\substack{|A|=m\\ |\varepsilon_j|=1}} \left\| \sum_{j\in A} \varepsilon_j \mathbf{x}_j \right\|, \quad m \in \mathbb{N}.$$

A standard convexity argument yields

$$\varphi_m[\mathcal{B}, \mathbb{X}] \le \sup_{\substack{|A|=m\\ |a_j|\le 1}} \left\| \sum_{j\in A} a_j \mathbf{x}_j \right\| = \varphi_m^\varepsilon[\mathcal{B}, \mathbb{X}] \le \kappa\, \varphi_m[\mathcal{B}, \mathbb{X}]. \tag{2.1}$$

If $\mathbb{Y}$ is the subspace of $\mathbb{X}^*$ spanned by $\mathcal{B}^*$, we set

$$\varphi_m^{\varepsilon,*}[\mathcal{B}, \mathbb{X}] = \varphi_m^{\varepsilon,*} := \varphi_m^\varepsilon[\mathcal{B}^*, \mathbb{Y}], \qquad m \in \mathbb{N},$$

and define the sequence

$$\mathbf{B}_m[\mathcal{B}, \mathbb{X}] = \mathbf{B}_m := \sup_{r\le m} \frac{\varphi_r^\varepsilon[\mathcal{B}, \mathbb{X}]\, \varphi_r^{\varepsilon,*}[\mathcal{B}, \mathbb{X}]}{r}.$$

Then a basis $\mathcal{B}$ is bi-democratic if and only

$$\Delta_{sb}[\mathcal{B}, \mathbb{X}] := \sup_m \mathbf{B}_m < \infty. \tag{2.2}$$

Quantitatively we have

$$\Delta_b[\mathcal{B}, \mathbb{X}] \le \Delta_{sb}[\mathcal{B}, \mathbb{X}] \le \kappa^2\, \Delta_b[\mathcal{B}, \mathbb{X}]. \tag{2.3}$$

Remark 2.1. *The identity*

$$\left(\sum_{j\in A} \mathbf{x}_j^*\right)\left(\sum_{j\in A} \mathbf{x}_j\right) = |A|, \quad A \subseteq J,\ |A| < \infty$$

yields (see [6]*)*

$$\Delta[\mathcal{B}, \mathbb{X}] \le \Delta_b[\mathcal{B}, \mathbb{X}]$$

Similarly, the identity

$$\left(\sum_{j\in A} \varepsilon_n \, \mathbf{x}_j^*\right) \left(\sum_{j\in A} \overline{\varepsilon_n} \, \mathbf{x}_j\right) = |A|, \quad A \subseteq J, \, |A| < \infty, \, |\varepsilon_j| = 1$$

gives

$$\Delta_s[\mathcal{B}, \mathbb{X}] \le \Delta_{sb}[\mathcal{B}, \mathbb{X}].$$

Garrigós et al. [8] gave several estimates for the constants $\mathbf{L}_m[\mathcal{B}, \mathbb{X}]$ involving the sequences of democracy constants and of conditionality constants of the basis. Subsequently, Berná et al. [4] obtained estimates for $\mathbf{L}_m[\mathcal{B}, \mathbb{X}]$ that also involved the sequence of quasi-greedy constants of the basis. The estimate we shall provide involves the sequence of bi-democracy constants and the sequence of conditionality constants $(\mathbf{k}_m^c)_{m=1}^\infty$ given by

$$\mathbf{k}_m^c[\mathcal{B}, \mathbb{X}] = \mathbf{k}_m^c := \sup_{|A| \le m} \|\mathrm{Id}_{\mathbb{X}} - S_A[\mathcal{B}, \mathbb{X}]\|.$$

Note that by (1.4), a basis $\mathcal{B}$ is unconditional if and only if

$$\sup_m \mathbf{k}_m^c = K_{su}[\mathcal{B}, \mathbb{X}] < \infty.$$

Lemma 2.2. *Let $\mathcal{B} = (\mathbf{x}_j)_{j\in J}$ be a semi-normalized M-bounded basis in a Banach space $\mathbb{X}$. For every $f \in \mathbb{X}$ and $m \in \mathbb{N}$,*

$$a_m^*[\mathcal{B}, \mathbb{X}](f) \, \varphi_m^\varepsilon[\mathcal{B}, \mathbb{X}] \le \mathbf{B}_m[\mathcal{B}, \mathbb{X}] \, \|f\|.$$

Proof. Let $G \subseteq J$ be such that $|G| = m$ and $a_m^* \le |\mathbf{x}_j^*(f)|$ for all $j \in G$. Define $f^* \in \mathbb{X}^*$ by $f^* = \sum_{j\in G} \overline{\operatorname{sign} \mathbf{x}_j^*(f)} \, \mathbf{x}_j^*$. Then

$$a_m^*(f) \varphi_m^\varepsilon \le \mathbf{B}_m \frac{m \, a_m^*(f)}{\varphi_m^{\epsilon,*}} \le \mathbf{B}_m \frac{\sum_{j\in G} |\mathbf{x}_j^*(f)|}{\|f^*\|} = \mathbf{B}_m \frac{f^*(f)}{\|f^*\|} \le \mathbf{B}_m \|f\|.$$

□

Theorem 2.3. *Let $\mathcal{B} = (\mathbf{x}_j)_{j\in J}$ be a semi-normalized M-bounded basis in a Banach space $\mathbb{X}$. For all $m \in \mathbb{N}$ we have*

$$\mathbf{L}_m[\mathcal{B}, \mathbb{X}] \le \mathbf{k}_{2m}^c[\mathcal{B}, \mathbb{X}] + \mathbf{B}_m[\mathcal{B}, \mathbb{X}].$$

Proof. Let $f \in \mathbb{X}$, $m \in \mathbb{N}$ and $G \subseteq J$ of cardinality m such that $\mathcal{G}_m(f) = S_G(f)$. Let $A \subseteq J$ with $|A| = m$ and $(a_j)_{j\in A} \in \mathbb{F}^A$. Put $g = f - \sum_{j\in A} a_j \, \mathbf{x}_j$. We have

$$\|f - \mathcal{G}_m(f)\| = \|g - S_{A\cup G}(g) + S_{A\setminus G}(f)\| \le \|g - S_{A\cup G}(g)\| + \|S_{A\setminus G}(f)\|.$$

Since $|A \cup B| \le 2m$,

$$\|g - S_{A\cup G}(g)\| \le \mathbf{k}_{2m}^c \, \|g\|.$$

Let $r = |A \setminus G| = |G \setminus A|$. Invoking (2.1) and Lemma 2.2,

$$\begin{aligned}\|S_{A\setminus G}(f)\| &\le \max_{j\in A\setminus G} |\mathbf{x}_j^*(f)|\varphi_r^\varepsilon \\ &\le \min_{j\in G\setminus A} |\mathbf{x}_j^*(f)|\varphi_r^\varepsilon \\ &= \min_{j\in G\setminus A} |\mathbf{x}_j^*(g)|\varphi_r^\varepsilon \\ &\le a_r^*(g)\varphi_r^\varepsilon \\ &\le \mathbf{B}_r \, \|g\|.\end{aligned}$$

Taking into account that $r \le m$ and that $(\mathbf{B}_m)_{m=1}^\infty$ is non-decreasing we get

$$\|S_{A\setminus G}(f)\| \le \mathbf{B}_m \, \|g\|.$$

Combining we obtain the desired result. □

Proof of Theorem 1.4. Taking the supremum over m in Theorem 2.3 and appealing to (2.3) gives

$$C_g[\mathcal{B},\mathbb{X}] \le K_{su}[\mathcal{B},\mathbb{X}] + \Delta_{sb}[\mathcal{B},\mathbb{X}] \le K_{su}[\mathcal{B},\mathbb{X}] + \kappa^2\, \Delta_b[\mathcal{B},\mathbb{X}]. \tag{2.4}$$

□

3. Estimates for the Haar basis in L_p

We start this section with the proof advertised in Section 1 of the lower estimate for the democracy constant.

Proof of Proposition 1.2. If $(J_j)_{j=1}^m$ are disjointly supported intervals in $\mathcal{D}$ we have

$$\left\| \sum_{j=1}^m h_{J_j}^{(p)} \right\|_p = m^{1/p}. \tag{3.1}$$

Let $(I_j)_{j=1}^\infty$ be the sequence in $\mathcal{D}$ defined recursively as follows: $I_1 = [0,1)$ and I_{j+1} is the left half of I_j. Set $q = p'$. Then

$$\begin{aligned}\left\| \sum_{j=1}^m h_{I_j}^{(p)} \right\|_p^p &= 2^{-m} \left| \sum_{k=0}^m 2^{k/p} \right|^p + \sum_{j=0}^{m-1} 2^{-j-1} \left| 2^{j/p} - \sum_{k=0}^{j-1} 2^{k/p} \right|^p \\ &= \frac{(2^{1/p} - 2^{-m/p})^p + \sum_{j=0}^{m-1} |(2^{1/q} - 1 - 2^{-(j+1)/p}|^p}{(2^{1/p}-1)^p} \\ &= \frac{\|f-g\|_{\ell_p}^p}{(2^{1/p}-1)^p},\end{aligned}$$

where

$$f = (\underbrace{2^{1/q}-1, \dots, 2^{1/q}-1}_{m \text{ times}}, 2^{1/p}), \quad g = (2^{-1/p}, \dots, 2^{-j/p}, \dots, 2^{-m/p}, 2^{-m/p}).$$

We have $\|g\|_{\ell_p} = 1$ and $\|f\|_{\ell_p} = (2 + m(2^{1/q} - 1)^p)^{1/p}$. Hence, by Minkowski's inequality,

$$\frac{(2+m(2^{1/q}-1)^p)^{1/p}-1}{2^{1/p}-1} \le \left\| \sum_{j=1}^m h_{I_j}^{(p)} \right\|_p \le \frac{(2+m(2^{1/q}-1)^p)^{1/p}+1}{2^{1/p}-1}. \tag{3.2}$$

Comparing (3.1) with (3.2) and letting m tend to ∞ we get

$$\Delta_d[\mathcal{H}^{(p)}, L_p] \ge \max\left\{ \frac{2^{1/q}-1}{2^{1/p}-1}, \frac{2^{1/p}-1}{2^{1/q}-1} \right\},$$

as desired. □

Next we establish the upper estimate for the super-bi-democracy constants that we will need.

Proposition 3.1. *Let $1 < p < \infty$. Then*

$$\Delta_{sb}[\mathcal{H}^{(p)}, L_p] \le D_p := \frac{8}{(2^{1/p}-1)(2^{1/p'}-1)}.$$

Proof. For $I \in \mathcal{D}$ let $n(I)$ be such that $|I| = 2^{-n(I)}$. Let $A \subseteq \mathcal{D}$ finite and $\varepsilon = (\varepsilon_I)_{I\in A}$ be such that $|\varepsilon_I| = 1$ for all $I \in A$. For $J \in A$ set

$$R_J = J \setminus \cup\{I \colon I \in A, n(I) > n(J)\}.$$

Taking into account that, for $n \in \mathbb{N}$, the collection of dyadic intervals $\{I \in \mathcal{D} \colon n(I) = n\}$ is a partition on $[0, 1)$ we infer that

- $(R_I)_{I\in A}$ is a partition of $K = \cup_{I\in A} I$,
- $R_I \subseteq I$ for every $I \in A$, and
- given $t \in K$ and $k \in \mathbb{N}$ there is at most one interval $I_{t,k} \in A \cap \mathcal{D}_k$ such that $t \in I_{t,k}$; moreover $n(I_{t,k}) \le n(J)$ if $t \in J$.

Consequently, for any $t \in K$,

$$\begin{aligned}
\left| \sum_{I\in A} \varepsilon_I h_I^{(p)}(t) \right| &\le \sum_{I\in A} \left| h_I^{(p)}(t) \right| \\
&= \sum_{I\in A} 2^{n(I)/p} \chi_I(t) \\
&\le \sum_{J\in A} \left(\sum_{n=-\infty}^{n(J)} 2^{n/p} \right) \chi_{R_J}(t) \\
&= \frac{1}{1-2^{-1/p}} \sum_{J\in A} 2^{n(J)/p} \chi_{R_J}(t).
\end{aligned}$$

Hence, if we set $a_p = 1/(1-2^{-1/p})$ we obtain

$$\begin{aligned}\left\|\sum_{I\in A}\varepsilon_I h_I^{(p)}\right\|_p &\le a_p\left(\sum_{J\in A}2^{n(J)}|R_J|\right)^{1/p}\\ &\le a_p\left(\sum_{J\in A}2^{n(J)}|J|\right)^{1/p}\\ &= a_p|A|^{1/p}.\end{aligned}$$

Therefore,

$$\left\|h_0^{(p)}+\sum_{I\in A}\varepsilon_I h_I^{(p)}\right\|_p \le 1+a_p|A|^{1/p}.$$

We infer that, for $m\in\mathbb{N}$,

$$\varphi_m^\varepsilon[\mathcal{H}^{(p)},L_p]\le \max\{a_p m^{1/p}, 1+a_p(m-1)^{1/p}\}\le 2a_p m^{1/p}.$$

The fact that $\varphi_m^{\varepsilon,*}[\mathcal{H}^{(p)},L_p]=\varphi_m^\varepsilon[\mathcal{H}^{(p')},L_{p'}]$ for $m\in\mathbb{N}$ yields

$$\frac{\varphi_m^\varepsilon[\mathcal{H}^{(p)},L_p]\,\varphi_m^{\varepsilon,*}[\mathcal{H}^{(p)},L_p]}{m}\le 4a_p a_{p'}\frac{m^{1/p}m^{1/p'}}{m}=4a_pa_{p'}=D_p.$$

Thus

$$\Delta_{sb}[\mathcal{H}^{(p)},L_p]\le D_p.$$

□

Corollary 3.2. *Let $1<p<\infty$. Then*

$$\begin{aligned}\Delta_b[\mathcal{H}^{(p)},L_p]&\approx\Delta_s[\mathcal{H}^{(p)},L_p]\approx\Delta_{sd}[\mathcal{H}^{(p)},L_p]\approx\Delta[\mathcal{H}^{(p)},L_p]\approx\Delta_d[\mathcal{H}^{(p)},L_p]\\ &\approx p^*.\end{aligned}$$

Proof. Let D_p and d_p be as in Proposition 1.2 and Propostion 3.1. We have $d_p\approx D_p\approx p^*$ for $1<p<\infty$. Then the result follows by combining Proposition 1.2, Propostion 3.1, and Remark 2.1. □

We are now in a position to complete the proof of our main theorem.

Conclusion of the Proof of Theorem 1.3. Combine Theorem 1.4 with the left-hand side of inequality (1.8), Corollary 3.2 and Theorem 1.1. □

Corollary 3.3. $C_a[\mathcal{H}^{(p)},L_p]\approx p^*$ *for* $1<p<\infty$.

Proof. Just combine the left-hand side inequality in (1.11) with (1.12), Corollary 3.2, and Theorem 1.3. □

Notice that an alternative way to formulate Theorem 1.3 is

$$0<\alpha:=\liminf_{p^*\to\infty}\frac{C_g[\mathcal{H}^{(p)},L_p]}{p^*}\le\beta:=\limsup_{p^*\to\infty}\frac{C_g[\mathcal{H}^{(p)},L_p]}{p^*}<\infty,$$

and so a subsequent natural task would be to compute (or estimate) the constants α and β. Let us record the estimates that the techniques developed in this note provide. On the one hand, combining the left-hand side of (1.8) with Proposition 1.2 we obtain

$$\alpha\ge\liminf_{p^*\to\infty}\frac{d_p}{p^*}=\frac{1}{\log 2}.$$

On the other hand, combining (2.4) with (1.6), Theorem 1.1 and Proposition 3.1 yields

$$\beta\le 1+\limsup_{p^*\to\infty}\frac{D_p}{p^*}=1+\frac{8}{\log 2}.$$

Other related questions are the following.

Question 3.4. *Set* $H(p)=C_g[\mathcal{H}^{(p)},L_p]$.

(a) *Is* H *decreasing on* $(1,2]$ *and increasing on* $[2,\infty)$*?*
(b) *Is* H *symmetric with respect to* $p=2$*? i.e., is,* $H(p)=H(p')$ *for* $p\in(1,\infty)$*?*

4. Appendix: Summary of the most commonly employed constants

Symbol	**Name of constant**	**Ref. equation**
C_a	Symmetry for largest coeffs. constant	(1.10)
C_g	Greedy constant	(1.3)
Δ	Democracy constant	(1.7)
Δ_b	Bi-democracy constant	(1.13)
Δ_d	Disjoint-democracy constant	(1.7)
Δ_s	Superdemocracy constant	(1.9)
Δ_{sb}	Super bi-democratic constant	(2.2)
Δ_{sd}	Disjoint-superdemocracy constant	(1.9)
K_{su}	Suppression unconditional constant	(1.4)
K_u	Lattice unconditional constant	(1.5)

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